Gas Dynamics Equations Summary

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Abstract

This document is a summary of the equations that appeared in the book “Fundamentals of compressible Flow Mechanics.” This summary supposed to be used by professionals and students who would like to have a handy summary of the equations without going through the pages of the whole books.

1 Introduction

Many have asked me to make a summary for the working questions for the gas dynamics. Due to time constrains, this document wasn't constructed. Therefore, I found myself searching for an equation in the book and I realized the importance and the urgency of this document. At this stage, this collection is a “quick–fix” which will be improved, hopefully, in the coming days. These equations were collected from the book “Fundamentals of compressible Flow” by Genick Bar-Meir and translated by using latex2html versions 1.7.

2 Speed of Sound

2.1 General

The general equation of speed sound is

$$\frac{dP}{d\rho} = \left.\frac{\partial P}{\partial \rho}\right|_{s} \tag{1}$$

2.2 Ideal Gas

Gas that obey the equation of state $P = \rho RT$, the speed of sound is

$$c = \sqrt{kRT} \tag{2}$$

Gas that obey the equation of state $P = z\rho RT$, the speed of sound is

$$c = \sqrt{nzRT} \tag{3}$$

Where n is defined as

$$n = \overbrace{\frac{C_p}{C_v}}^{k} \left(\frac{z + T \left(\frac{\partial z}{\partial T}\right)_\rho}{z + T \left(\frac{\partial z}{\partial T}\right)_P} \right) \tag{4}$$

2.3 Speed of Sound in Liquid

$$c = \sqrt{\frac{elastic\ property}{inertial\ property}} = \sqrt{\frac{B}{\rho}} \tag{5}$$

where

$$B = \rho \frac{dP}{d\rho} \tag{6}$$

2.4 Speed of Sound in Solids

$$c = \sqrt{\frac{E}{\rho}} \tag{7}$$

where E is Young's Modulus

2.5 Sound Speed in Two Phase Medium

For flow of mostly gas with drops of the other phase (liquid or solid)

Let

$$\frac{\rho}{\rho_a} = 1 + m \tag{8}$$

where $m = \frac{\dot{m}_b}{\dot{m}_a}$ is mass flow rate per gas flow rate. and the subscript a is for the gas phase and b for the liquid or solid phase.

The equation of state is

$$\frac{P}{\rho} = \frac{R}{1+m} T \tag{9}$$

$$c = \sqrt{\gamma R_{mix} T} \tag{10}$$

where

$$\gamma = \frac{C_p + mC}{C_v + mC} \tag{11}$$

and $R_{mix} = \frac{R}{1+m}$

3 Isentropic Flow

$$\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{\frac{k}{k-1}} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} \tag{12}$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{k-1}} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{1}{k-1}} \tag{13}$$

The star values

$$\frac{T^*}{T_0} = \frac{{c^*}^2}{{c_0}^2} = \frac{2}{k+1} \tag{14}$$

$$\frac{P^*}{P_0} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \tag{15}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \tag{16}$$

3.1 Relationships for Small Mach Number

$$\frac{P_0}{P} = 1 + \frac{(k-1)M^2}{4} + \frac{kM^4}{8} + \frac{2(2-k)M^6}{48} \cdots \tag{17}$$

$$\frac{\rho_0}{\rho} = 1 + \frac{(k-1)M^2}{4} + \frac{kM^4}{8} + \frac{2(2-k)M^6}{48} \cdots \tag{18}$$

$$\frac{P_0 - P}{\frac{1}{2}\rho U^2} = 1 + \overbrace{\frac{M^2}{4} + \frac{(2-k)M^4}{24} + \cdots}^{\textit{compressibility correction}} \tag{19}$$

$$M^* = \frac{U}{c^*} = \sqrt{\frac{k+1}{2}} M \left(1 - \frac{k-1}{4} M^2 + \cdots\right) \tag{20}$$

$$\frac{P_0 - P}{P} = \frac{kM^2}{2}\left(1 + \frac{M^2}{4} + \cdots\right) \tag{21}$$

$$\frac{\rho_0 - \rho}{\rho} = \frac{M^2}{2}\left(1 - \frac{kM^2}{4} + \cdots\right) \tag{22}$$

$$\frac{\dot{m}}{A} = \sqrt{\frac{k{P_0}^2M^2}{RT_0}}\left(1 + \frac{k-1}{4}M^2 + \cdots\right) \tag{23}$$

The ratio of the area to star area is

$$\frac{A}{A^*} = \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}}\left(\frac{1}{M} + \frac{k+1}{4}M + \frac{(3-k)(k+1)}{32}M^3 + \cdots\right) \tag{24}$$

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}}\right)^{\frac{k+1}{2(k-1)}} \tag{25}$$

3.2 Isentropic Isothermal Flow Nozzle

$$T_1 = T_2 \tag{26}$$

$$\frac{T_{01}}{T_{02}} = \frac{\left(1 + \frac{k-1}{2}{M_1}^2\right)}{\left(1 + \frac{k-1}{2}{M_2}^2\right)} = \frac{\left(1 + \frac{k-1}{2}{M_1}^2\right)}{\left(1 + \frac{k-1}{2}{M_2}^2\right)} \tag{27}$$

$$\frac{P_2}{P_1} = e^{\frac{k({M_1}^2 - {M_2}^2)}{2}} = \left(\frac{e^{{M_1}^2}}{e^{{M_2}^2}}\right)^{\frac{k}{2}} \tag{28}$$

$$\frac{A_2}{A_1} = \frac{M_1}{M_2}\left(\frac{\mathrm{e}^{{M_2}^2}}{\mathrm{e}^{{M_1}^2}}\right)^{\frac{k}{2}} \tag{29}$$

$$\frac{P_{02}}{P_{01}} = \frac{P_2}{P_1}\left(\frac{1 + \frac{k-1}{2}{M_2}^2}{1 + \frac{k-1}{2}{M_1}^2}\right)^{\frac{k}{k-1}} = \left[\frac{\mathrm{e}^{{M_1}^2}}{\mathrm{e}^{{M_1}^2}}\right]^{\frac{k}{2}} \tag{30}$$

The star values

$$T = T^* \tag{31}$$

$$\frac{P}{P^*} = \frac{\rho}{\rho^*} = \mathrm{e}^{\frac{(1-M^2)k}{2}} \tag{32}$$

$$\frac{A}{A^*} = \frac{1}{M}\mathrm{e}^{\frac{(1-M^2)k}{2}} \tag{33}$$

$$\frac{T_0}{{T_0}^*} = \frac{2\left(1+\frac{k-1}{2}{M_1}^2\right)}{k+1}^{\frac{k}{k-1}} \tag{34}$$

$$\frac{P_0}{{P_0}^*} = e^{\frac{(1-M)k}{2}}\frac{2\left(1+\frac{k-1}{2}{M_1}^2\right)}{k+1}^{\frac{k}{k-1}} \tag{35}$$

The initial stagnation temperature is denoted as T_{0int}.

$$\frac{T}{T_{0int}} = \frac{1}{1+\frac{k-1}{2}M^2} \tag{36}$$

$$\frac{P}{P_{0int}} = \frac{1}{\left(1+\frac{k-1}{2}M^2\right)^{\frac{k-1}{k}}} \tag{37}$$

$$\frac{F_{net}}{P_0 A^*} = \overbrace{\frac{P_2 A_2}{P_0 A^*}}^{f(M_2)}\overbrace{\left(1+k{M_2}^2\right)}^{f(M_2)} - \overbrace{\frac{P_1 A_1}{P_0 A^*}}^{f(M_1)}\overbrace{\left(1+k{M_1}^2\right)}^{f(M_1)} \tag{38}$$

$$\frac{F}{F^*} = \frac{P_1 A_1}{P^* A^*} \frac{\left(1 + k{M_1}^2\right)}{(1+k)} = \frac{1}{\underbrace{\frac{P^*}{P_0}}_{\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}}} \overbrace{\frac{P_1 A_1}{P_0 A^*}\left(1 + k{M_1}^2\right)}^{\text{see function (38)}} \frac{1}{(1+k)} \tag{39}$$

$$F_{net} = P_0 A^* (1+k) \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} \left(\frac{F_2}{F^*} - \frac{F_1}{F^*}\right) \tag{40}$$

for isothermal

$$\frac{F_2}{F_1} = \frac{P_2 A_2}{P_1 A_1} \frac{1 + \frac{{U_2}^2}{RT}}{1 + \frac{{U_1}^2}{RT}} \tag{41}$$

$$\frac{F_2}{F_1} = \frac{M_1}{M_2} \frac{1 + k{M_2}^2}{1 + k{M_1}^2} \tag{42}$$

$$\frac{F_2}{F^*} = \frac{1}{M_2} \frac{1 + k{M_2}^2}{1 + k} \tag{43}$$

4 Normal Shock

$$T_{0y} = T_{0x} \tag{44}$$

$$\frac{T_y}{T_x} = \left(\frac{P_y}{P_x}\right)^2 \left(\frac{M_y}{M_x}\right)^2 \tag{45}$$

$$\frac{P_y}{P_x} = \frac{1 + k{M_x}^2}{1 + k{M_y}^2} \tag{46}$$

$$\frac{P_{0y}}{P_{0x}} = \frac{P_y \left(1 + \frac{k-1}{2}{M_y}^2\right)^{\frac{k}{k-1}}}{P_x \left(1 + \frac{k-1}{2}{M_x}^2\right)^{\frac{k}{k-1}}} \tag{47}$$

$${M_y}^2 = \frac{{M_x}^2 + \frac{2}{k-1}}{\frac{2k}{k-1}{M_x}^2 - 1} \tag{48}$$

$$\begin{aligned} \frac{P_y}{P_x} &= \frac{2k}{k+1}{M_x}^2 - \frac{k-1}{k+1} \\ \frac{P_y}{P_x} &= 1 + \frac{2k}{k+1}\left({M_x}^2 - 1\right) \end{aligned} \tag{49}$$

$$\frac{\rho_y}{\rho_x} = \frac{U_x}{U_y} = \frac{(k+1){M_x}^2}{2 + (k-1){M_x}^2} \tag{50}$$

$$\frac{T_y}{T_x} = \left(\frac{P_y}{P_x}\right)\left(\frac{\frac{k+1}{k-1} + \frac{P_y}{P_x}}{1 + \frac{k+1}{k-1}\frac{P_y}{P_x}}\right) \tag{51}$$

$$\frac{\rho_x}{\rho_y} = \frac{1 + \left(\frac{k+1}{k-1}\right)\left(\frac{P_y}{P_x}\right)}{\left(\frac{k+1}{k-1}\right) + \left(\frac{P_y}{P_x}\right)} \tag{52}$$

Moving shocks

5 Isothermal Flow

$$\int_0^L \frac{4f dx}{D} = \int_{M^2}^{1/k} \frac{1 - kM2}{kM2} dM^2 \tag{53}$$

$$\tfrac{4fL_{max}}{D} = \frac{1 - kM^2}{kM^2} + \ln kM^2 \tag{54}$$

$$\frac{P_0}{P_0^*} = \frac{P}{P^*} \left[\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2k}} \right]^{\frac{k}{k-1}} \tag{55}$$

$$\frac{P_0}{P_0^*} = \frac{1}{\sqrt{k}} \left(\frac{2k}{3k - 1} \right)^{\frac{k}{k-1}} \left(1 + \frac{k-1}{2} M^2 \right)^{\frac{k}{k-1}} \frac{1}{M} \tag{56}$$

$$\frac{T_0}{T_0^*} = \frac{T}{T^*} \frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2k}} = \frac{2k}{3k - 1} \left(1 + \frac{k-1}{2} \right) M^2 \tag{57}$$

$$\tfrac{4fL}{D} = \tfrac{4fL_{max}}{D}\Big|_1 - \tfrac{4fL_{max}}{D}\Big|_2 = \frac{1 - k{M_1}^2}{k{M_1}^2} - \frac{1 - k{M_2}^2}{k{M_2}^2} + \ln \left(\frac{M_1}{M_2} \right)^2 \tag{58}$$

For the case that $M_1 >> M_2$ and $M_1 \to 1$ equation (58) is reduced into the following approximation

$$\tfrac{4fL}{D} = 2 \ln M_1 - 1 - \overbrace{\frac{1 - k{M_2}^2}{k{M_2}^2}}^{\sim 0} \tag{59}$$

$$M_1 \sim \mathrm{e}^{\frac{1}{2}\left(\frac{4fL}{D} + 1\right)} \tag{60}$$

6 Fanno Flow

$$\frac{4f dx}{D} = \frac{\left(1 - M^2\right) dM^2}{kM^4 (1 + \frac{k-1}{2} M^2)} \tag{61}$$

$$\frac{4}{D} \int_L^{L_{max}} f dx = \frac{1}{k} \frac{1 - M^2}{M^2} + \frac{k+1}{2k} \ln \frac{\frac{k+1}{2} M^2}{1 + \frac{k-1}{2} M^2} \tag{62}$$

A representative friction factor is defined as

$$\bar{f} = \frac{1}{L_{max}} \int_0^{L_{max}} f dx \tag{63}$$

$$\frac{4\bar{f} L_{max}}{D} = \frac{1}{k} \frac{1 - M^2}{M^2} + \frac{k+1}{2k} \ln \frac{\frac{k+1}{2} M^2}{1 + \frac{k-1}{2} M^2} \tag{64}$$

$$\frac{P}{P^*} = \frac{1}{M} \sqrt{\frac{\frac{k+1}{2}}{1 + \frac{k-1}{2} M^2}} \tag{65}$$

$$\frac{T}{T^*} = \frac{c^2}{{c^*}^2} = \frac{\frac{k+1}{2}}{1 + \frac{k-1}{2} M^2} \tag{66}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M} \sqrt{\frac{1 + \frac{k-1}{2} M^2}{\frac{k+1}{2}}} \tag{67}$$

$$\frac{U}{U^*} = \left(\frac{\rho}{\rho^*}\right)^{-1} = M \sqrt{\frac{\frac{k+1}{2}}{1 + \frac{k-1}{2} M^2}} \tag{68}$$

$$\frac{P_0}{{P_0}^*} = \frac{1}{M}\left(\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}\right)^{\frac{k+1}{2(k-1)}} \tag{69}$$

$$\frac{s-s^*}{c_p} = \ln M^2\sqrt{\left(\frac{k+1}{2M^2\left(1+\frac{k-1}{2}M^2\right)}\right)^{\frac{k+1}{k}}} \tag{70}$$

$$\frac{T_2}{T_1} = \frac{\frac{T}{T^*}\Big|_{M_2}}{\frac{T}{T^*}\Big|_{M_1}} \tag{71}$$

$$\left(\frac{4fL_{max}}{D}\right)_2 = \left(\frac{4fL_{max}}{D}\right)_1 - \frac{4fL}{D} \tag{72}$$

7 RAYLEIGH FLOW

$$\frac{P^*}{P_1} = \frac{1+k{M_1}^2}{1+k} \tag{73}$$

$$\frac{T^*}{T_1} = \frac{1}{M^2}\left(\frac{1+k{M_1}^2}{1+k}\right)^2 \tag{74}$$

$$\frac{\rho_1}{\rho^*} = \frac{U^*}{U_1} = \frac{\frac{U^*}{\sqrt{kRT^*}}\sqrt{kRT^*}}{\frac{U_1}{\sqrt{kRT_1}}\sqrt{kRT_1}} = \frac{1}{M_1}\sqrt{\frac{T^*}{T_1}} \tag{75}$$

$$\frac{T_{01}}{{T_0}^*} = \frac{T_1\left(1+\frac{k-1}{2}{M_1}^2\right)}{T^*\left(\frac{1+k}{2}\right)} = \frac{2(1+k){M_1}^2}{(1+kM^2)^2}\left(1+\frac{k-1}{2}{M_1}^2\right) \tag{76}$$

$$\frac{P_{01}}{{P_0}^*} = \frac{P_1\left(1+\frac{k-1}{2}{M_1}^2\right)}{P^*\left(\frac{1+k}{2}\right)} = \left(\frac{1+k}{1+k{M_1}^2}\right)\left(\frac{1+k{M_1}^2}{\frac{(1+k)}{2}}\right)^{\frac{k}{k-1}} \tag{77}$$

8 Oblique-Shock

$$\tan\theta = \frac{U_{1n}}{U_{1t}} \tag{78}$$

$$\tan(\theta-\delta) = \frac{U_{2n}}{U_{2t}} \tag{79}$$

$$\sin\theta = \frac{M_{1n}}{M_1} \tag{80}$$

$$\sin(\theta-\delta) = \frac{M_{2n}}{M_2} \tag{81}$$

$$\cos\theta = \frac{M_{1t}}{M_1} \tag{82}$$

$$\cos(\theta-\delta) = \frac{M_{2t}}{M_2} \tag{83}$$

$$\tan\delta = 2\cot\theta\left[\frac{{M_1}^2\sin^2\theta-1}{{M_1}^2\left(k+\cos 2\theta\right)+2}\right] \tag{84}$$

$$\frac{\rho_2}{\rho_1} = \frac{U_{1n}}{U_{2n}} = \frac{(k+1){M_1}^2\sin^2\theta}{(k-1){M_1}^2\sin^2\theta+2} \tag{85}$$

$$\frac{T_2}{T_1} = \frac{2k{M_1}^2\sin^2\theta-(k-1)\left[(k-1){M_1}^2+2\right]}{(k+1)^2 M_1} \tag{86}$$

The Rankine–Hugoniot relations are the same as the relationship for the normal shock

$$\frac{P_2 - P_1}{\rho_2 - \rho_1} = k\frac{P_2 - P_1}{\rho_2 - \rho_1} \tag{87}$$

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0 \tag{88}$$

where

$$x = \sin^2\theta \tag{89}$$

and

$$a_1 = -\frac{{M_1}^2 + 2}{{M_1}^2} - k\sin^2\delta \tag{90}$$

$$a_2 = -\frac{2{M_1}^2 + 1}{{M_1}^4} + \left[\frac{(k+1)^2}{4} + \frac{k-1}{{M_1}^2}\right]\sin^2\delta \tag{91}$$

$$a_3 = -\frac{\cos^2\delta}{{M_1}^4} \tag{92}$$

$$x_1 = -\frac{1}{3}a_1 + (S + T) \tag{93}$$

$$x_2 = -\frac{1}{3}a_1 - \tfrac{1}{2}(S+T) + \tfrac{1}{2}i\sqrt{3}(S-T) \tag{94}$$

and

$$x_3 = -\frac{1}{3}a_1 - \tfrac{1}{2}(S+T) - \tfrac{1}{2}i\sqrt{3}(S-T) \tag{95}$$

Where

$$S = \sqrt[3]{R + \sqrt{D}}, \tag{96}$$

$$T = \sqrt[3]{R - \sqrt{D}} \tag{97}$$

and where the definition of the D is

$$D = Q^3 + R^2 \tag{98}$$

and where the definitions of Q and R are

$$Q = \frac{3a_2 - {a_1}^2}{9} \tag{99}$$

and

$$R = \frac{9a_1a_2 - 27a_3 - 2{a_1}^3}{54} \tag{100}$$

$$\sin^2\theta_{max} = \frac{-1 + \frac{k+1}{4}{M_1}^2 + \sqrt{(k+1)\left[1 + \frac{k-1}{2}{M_1}^2 + \left(\frac{k+1}{2}M_1\right)^4\right]}}{k{M_1}^2} \tag{101}$$

A simplified case of the Maximum Deflection Mach Number's equation for large Mach number becomes

$$M_{1n} = \sqrt{\frac{k+1}{2k}}M_1 \quad \text{for} \quad M_1 >> 1 \tag{102}$$

$$M_{1n} = \frac{\sqrt{(k+1){M_1}^2 + 1 + \sqrt{({M_1}^2\left[{M_1}^2(k+1)^2 + 8(k^2-1)\right] + 16(1+k)}}}{2\sqrt{k}} \tag{103}$$

$$\frac{P_2}{P_1} = \frac{2k{M_1}^2\sin^2\theta - (k-1)}{k+1} \tag{104}$$

The density ratio can be expressed as

$$\frac{\rho_2}{\rho_1} = \frac{U_{1n}}{U_{2n}} = \frac{(k+1){M_1}^2\sin^2\theta}{(k-1){M_1}^2\sin^2\theta + 2} \tag{105}$$

$$\frac{T_2}{T_1} = \frac{{c_2}^2}{{c_1}^2} = \frac{\left(2k{M_1}^2 \sin^2\theta - (k-1)\right)\left((k-1){M_1}^2 \sin^2\theta + 2\right)}{(k+1){M_1}^2 \sin^2\theta} \tag{106}$$

$$ {M_2}^2 = \frac{(k+1)^2 {M_1}^4 \sin^2\theta - 4({M_1}^2 \sin^2\theta - 1)(k{M_1}^2 \sin^2\theta + 1)}{\left(2k{M_1}^2 \sin^2\theta - (k-1)\right)\left((k-1){M_1}^2 \sin^2\theta + 2\right)} \tag{107}$$

The ratio of the total pressure can be expressed as

$$\frac{P_{0_2}}{P_{0_1}} = \left[\frac{(k+1){M_1}^2 \sin^2\theta}{(k-1){M_1}^2 \sin^2\theta + 2}\right]^{\frac{k}{k-1}} \left[\frac{k+1}{2k{M_1}^2 \sin^2\theta - (k-1)}\right]^{\frac{1}{k-1}} \tag{108}$$

8.1 Given Two Angles, δ and θ

$${M_1}^2 = \frac{2(\cot\theta + \tan\delta)}{\sin 2\theta - (\tan\delta)(k + \cos 2\theta)} \tag{109}$$

$$\frac{2(P_2 - P_1)}{\rho U^2} = \frac{2\sin\theta\sin\delta}{\cos(\theta - \delta)} \tag{110}$$

$$\frac{\rho_2 - \rho_1}{\rho_2} = \frac{\sin\delta}{\sin\theta\cos(\theta - \delta)} \tag{111}$$

9 Prandtl-Meyer Function

$$\nu(M) = \theta(M) - \theta(M_{starting}) \tag{112}$$

$$= \sqrt{\frac{k+1}{k-1}} \tan^{-1}\left(\sqrt{\frac{k-1}{k+1}}\sqrt{M^2 - 1}\right) - \tan^{-1}\sqrt{M^2 - 1} \tag{113}$$

$$\nu_\infty = \frac{\pi}{2}\left[\sqrt{\frac{k+1}{k-1}} - 1\right] \tag{114}$$